A Laplace Transform Cookbook
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Abstract

AC circuit analysis may be conducted in the time domain with differential equations or in the so-called complex frequency domain. It turns out that many problems are greatly simplified when converted to the complex frequency domain. For example, integration and differentiation in the time domain become simple algebraic expressions in the complex frequency domain.

The Laplace transform converts a problem between these two domains.

Oliver Heaviside, an English engineer, originated much of this technique. When criticized for his lack of mathematical rigour, he responded with words to the effect that ‘one need not understand the process of digestion in order to eat’. In that spirit, we provide here a cookbook approach to the application of Laplace transforms.

In this paper, we show how concepts of the Laplace transform may be applied to electronic circuit analysis. We also show measurement examples in which Syscomp instruments are used to demonstrate theoretical results.

This document is best read on a video screen using a pdf viewer program (such as acroread) at 100% magnification. A monochrome print is useable but loses colour and some detail in the images.

My colleagues Glen Martinson and John Foster contributed significantly to this paper: Glen by first showing me how to use the Laplace transform, and John with his very thorough review of the document. (Of course, any errors remain my responsibility.)

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1 Introduction

Time is valuable in the practice of Engineering, and Engineers value tools which facilitate getting to an answer and a final design. The slide rule and nomograph are early examples. The spreadsheet, simulation program (such as Spice) and the Computer Algebra Program are more recent addition to the engineer’s toolkit.

In this paper, we’ll show a cookbook approach to the Laplace transform, using the Computer Algebra Program Maxima.

The Laplace transform is a technique for circuit analysis that facilitates the calculation of a network time response. An electrical network exists. A certain input signal is applied to the network. What is the output from the network as a function of time?

This type of calculation can be done entirely in the time domain, but it requires solving differential equations, which is challenging and time-consuming.

The Laplace transform converts the input signal and the differential equation describing the network, into algebraic expressions in what is known as the complex frequency domain. The output in the complex frequency domain is simply the product of the input and the network. The output of the network as a function of time is the Inverse Laplace transform of this product.

This is analogous to a frequency domain analysis of, say, a filter network. The input spectrum is multiplied by the frequency response of the filter to give the output spectrum.

The Laplace transform technique is a huge improvement over working directly with differential equations. It is relatively straightforward to convert an input signal and the network description into the Laplace domain. However, performing the Inverse Laplace transform can be challenging and require substantial work in algebra and calculus.

In this paper, we show how Maxima, a computer algebra program, can be used in pursuit of the forward and inverse Laplace transform.

1.1 Simulation and Analysis

We are often called upon to determine the magnitude and phase response or some other electrical properties of a particular network. Why do all the work to obtain a mathematical analysis when one could simply simulate the circuit?

For a circuit design where component values are known, the simulation has the advantage - it may be faster to do and it can take into account the imperfections of components like an operational amplifier. However, that is a specific solution and it doesn’t give full insight into the circuit behaviour.

You can change the simulation circuit values and determine their effect on the frequency and phase response. However, analysis gives a deeper understanding of the circuit operation. It can tell you for example that the cutoff frequency is proportional to the square root of the ratio of two capacitor values. Knowing that, you might be able to select two capacitors that track with temperature and therefore result in a more stable filter design.

A simulation gives circuit specific results when numeric values are known. Analysis provides generalized results. They both have their uses.

1.2 Case History

To set the stage for this next circuit and analysis, suppose that we wish to design a filter for noise pulses that occur on a power supply line, figure 1. We would like to attenuate this noise pulse sufficiently with as low cost a circuit as feasible.

The noise pulse can be modelled as an impulse, a pulse that is short compared to the time-constants in the circuit.

The filter candidates are shown in figure 2.

Each of these filters can be considered to be a voltage divider, in which the output voltage is some fraction of the input. The impedance of one or more components is

\[1\text{It can also be done using a technique called convolution. For many networks and signals, it turns out that convolution is not an attractive approach.}\]

\[2\text{The term magnitude is used to refer to the size of some behavioural property of an electrical component or network. The term amplitude, which we’ll meet later, is used to refer to the size of an electrical signal.}\]

\[3\text{In a final design, this vague description must be tightened up to specify the actual noise reduction and the resultant cost.}\]
variable with frequency. The net effect is to increase the voltage division ratio at high frequencies, that is, reduce the output signal at high frequencies. For example, in figure 2(a), the bottom arm of the voltage divider is a capacitor. At high frequencies, the impedance of the capacitor decreases and so also the output signal.

The single section RC filter is least expensive. The dual section RC filter requires 4 inexpensive components, but is a more effective filter. The RLC filter uses 3 components, one of which is a more expensive inductor. However, if the RLC filter has a significant advantage in performance, it might be preferable.

To do a meaningful comparison, we need to know the frequency response and the impulse response of each of these filters. First, some explanation of tools that can help get us there.

2 Laplace Notation as Shorthand

Very simply, the Laplace transform substitutes \( s \), the Laplace transform operator for the differential operator \( d/dt \). Then the \( s \) term may be manipulated like any other variable.

Thus one will see \( s \) in a control system block to indicate \emph{differentiator} and \( 1/s \) to indicate \emph{integrator}.

The substitution of \( s \) for \( d/dt \) leads to another one, \( s \) for \( j\omega \). This is useful in determining the transfer function of an electrical network, and then its magnitude and phase responses.

This approach can be used without any real understanding of Laplace transform theory. It can be regarded simply a kind of shorthand. Nonetheless, it’s a useful technique when the requirement is to determine the \emph{frequency response} of some network. In section 4 (page 17) we’ll look at obtaining the \emph{time-domain response}, which does require more in-depth understanding.

2.1 A Simple Example: Capacitor Charging Equation

As a first introduction to the Laplace operator \( s \) as differentiation and integration, consider the operation of a capacitor. The basic differential equation relating voltage and current in a capacitor is\(^4\):

\[
\frac{dv}{dt} = \frac{1}{C} \int i dt
\]

\(^4\)A Note on Notation: It is common to indicate time domain quantities as lower case such as \( v \) or \( v(t) \). We’ll use the latter. Common practice is also to use upper case for the Laplace domain (aka the \emph{complex frequency domain}), such as \( V \). However, sometimes upper case is used for DC quantities and lower case for AC quantities, so we will use \( v(s) \) for Laplace domain to make it very clear. British practice is to use \( p \) for \( s \).
\[ i(t) = C \frac{d}{dt} v(t) \]  

(1)

where:

- \( i(t) \) Current in the capacitor, amps, as a function of time
- \( v(t) \) Voltage across the capacitor, volts, as a function of time
- \( C \) Capacitance, farads

In words, the time-varying current into a capacitor is proportional to the rate of change of the voltage across its terminals. The constant of proportionality is the capacitance \( C \).

To apply the Laplace transform to this equation, we replace the differential operator \( \frac{d}{dt} \) by \( s \) and the voltage and current by their transformed versions:

\[ i(s) = C s v(s) \]

(2)

where:

- \( i(s) \) Current in the capacitor, amps, in the Laplace domain
- \( v(s) \) Voltage across the capacitor, volts, in the Laplace domain
- \( C \) Capacitance, farads

Let’s reverse this and solve for the capacitor voltage:

\[ v(s) = \frac{1}{C s} i(s) \]

(3)

Now, back to the time domain: voltage and current transform to their time-dependent values and \( 1/s \) becomes an integral:

\[ v(t) = \frac{1}{C} \int i(t) dt \]

(4)

This is the long way round. Equation 4 could be obtained directly by inspection of equation 1. However, it shows a very simple application of the Laplace transform: we transformed the original equation into the Laplace domain, manipulated it, and then transformed the result back into the time domain.

### 2.2 Inductor Differential Equation

A similar reasoning process can be applied to the inductor. The basic differential equation relating voltage and current in an inductor is:

\[ v(t) = L \frac{di(t)}{dt} \]

(5)

where:

- \( v(t) \) Voltage across the inductor, volts
- \( i(t) \) Current in the inductor, amps
- \( L \) Inductance, Henries

Proceeding as we did in the case of capacitance, we replace the differential operator \( d/dt \) by \( s \) and the voltage and current by their transformed versions:

\[ v(s) = L s i(s) \]

(6)

\[ = s L i(s) \]

*The result of equation 4 assumes that there is no initial charge on the capacitor. If there is charge on the capacitor at the beginning of the interval of integration, it must be accounted for by an additional term which we might call \( v_0 \).
The reactance of the inductor (analogous to resistance, but affecting AC current only) is the ratio of voltage to current:

\[
\frac{v(s)}{i(s)} = sL \tag{7}
\]

Inductors may then be represented by inductive reactance as \( sL_1, sL_2 \) and so on.

### 2.3 Capacitive Reactance

The reactance of the capacitor is the ratio of voltage to current:

\[
\frac{v(s)}{i(s)} = \frac{1}{sC} \tag{8}
\]

It's common to represent the capacitive reactance directly on a circuit diagram, so one sees capacitors labelled as \( 1/sC_1, 1/sC_2 \) and so on. For the purpose of circuit analysis these reactances may be treated as resistances.

The magnitude phase of capacitive reactance are also represented as

\[
X_c = \frac{1}{j\omega C} \tag{9}
\]

where the variables are:

- \( X_c \) Capacitive reactance, ohms
- \( j \) Imaginary operator, \( \sqrt{-1} \)
- \( \omega \) Circular frequency, radians/sec

Comparing equations 8 and 9, it can be seen that the Laplace transform operator \( s \) can be treated simply as shorthand notation for \( j\omega \).

### 2.4 Summary

- The differential operator \( d/dt \) in a differential equation can be replaced by \( s \). \(^6\)
- Integration is the inverse: \( 1/s \). \(^7\)
- The Laplace operator \( s \) may also be regarded as shorthand for the expression \( j\omega \).
- Capacitive reactance is \( 1/sC \).
- Inductive reactance is \( sL \).

\(^6\)This is straightforward to extend: double differentiation \( d^2/dt^2 \) becomes \( s^2 \), and so on.

\(^7\)Similarly, double integration becomes \( 1/s^2 \), and so on.
3 Transfer Function and Frequency Response

Next, we’ll develop the magnitude and phase response curves for the lowpass networks of figure 2.

3.1 Transfer Function of Low Pass RC Filter

A simple RC lowpass filter is shown in figure 4, where the capacitor is indicated by its reactance $1/sC$. Let us determine the frequency response of this filter.

The frequency response shows the relationship between output voltage and input voltage as a function of frequency. Consequently, a first step is to determine the relationship between $e_o$ and $e_i$. The resistor and the reactance of the capacitor form a voltage divider, so we can write:

$$\frac{e_o}{e_i} = \frac{Z_2}{Z_1 + Z_2}$$  \hspace{1cm} (10)

where:

- $e_o$: AC Output voltage voltage from the circuit
- $e_i$: AC Input voltage to the circuit
- $Z_1$: Impedance of the upper half of the voltage divider (the resistor)
- $Z_2$: Impedance of the bottom half of the voltage divider (the capacitor)

Strictly speaking, $e_o$ should be written as $e_o(\omega)$ or $e_o(f)$ to indicate that the value is a function of frequency, but we’ll take that as understood.

Now substitute $R$ for $Z_1$, $1/sC$ for $Z_2$ and do some algebra:

$$\frac{e_o}{e_i} = \frac{1/sC}{R + 1/sC} = \frac{1}{1 + sRC}$$  \hspace{1cm} (11)

Now we need to introduce some new labels. The quantity $RC$ is important in these circuits: it is known as the time constant $\tau$ and will turn up again when we look at the time-domain response of the filter.

$$\tau = RC$$  \hspace{1cm} (12)

Then we could rewrite equation 11 this way:

$$\frac{e_o}{e_i} = \frac{1}{1 + s\tau}$$  \hspace{1cm} (13)

We can do even better than this. In the frequency domain RC is related to the corner or cutoff frequency of the filter, which is referred to as $\omega_o$ in radians/sec notation or $f_o$ in Hertz (cycles/second).

$$\omega_o = \frac{1}{\tau} = \frac{1}{RC}$$  \hspace{1cm} (14)

So equation 13 could be written as:

$$\frac{e_o}{e_i} = \frac{1}{1 + s/\omega_o}$$  \hspace{1cm} (15)
This turns out to be a useful form of the equation as we’ll see in a second. It is known as the transfer function or characteristic equation of the RC lowpass network.

By the way, frequency in radians/second and Hertz are related

\[ \omega_o = 2\pi f_o \]  

(16)

So if you prefer your frequencies in Hz, you could rewrite equation 15 as:

\[ \frac{e_o}{e_i} = \frac{1}{1 + s/2\pi f_o} \]  

(17)

In other words, we have three ways to define the cutoff frequency: by time-constant \( \tau \), by cutoff frequency \( \omega_o \) in radians per second, and by cutoff frequency \( f_o \) in Hertz. All three are in use. Control system people like time constants. Electrical engineers use radians/second. Audio folks like Hertz.

We’ll stick with the form of equation 15.

3.1.1 Magnitude and Phase of The Lowpass RC Filter

![Amplitude and Phase of RC Lowpass Network](image)

Figure 5: Amplitude and Phase of RC Lowpass Network

Now we’ll put the transfer function of equation 15 into a suitable form for plotting the magnitude and phase response. To do that, we substitute \( j\omega \) for \( s \) in the transfer function:

\[ \frac{e_o}{e_i} = \frac{1}{1 + j\omega/\omega_o} \]  

(18)

Now, the denominator is a complex number with a real component 1 pointing along the X axis and a quadrature part \( \omega/\omega_o \) pointing up the Y axis. (See section 6.2 on page 35 for a review of complex numbers.)

We need to convert this complex number from its current rectangular form to polar form. In polar form, the magnitude is the hypotenuse of the two rectangular form vectors:

\[ \frac{e_o}{e_i} = \frac{1}{\sqrt{1 + (j\omega/\omega_o)^2}} \]

\[ = \frac{1}{\sqrt{1 - (\omega/\omega_o)^2}} \]  

(19)
The tangent of the phase angle is the ratio of the two rectangular components:

\[
\frac{e_0}{e_1} = \frac{1}{\tan^{-1} \left( \frac{\omega}{\omega_0} \right)}
\]

\[= -\tan^{-1} (\omega/\omega_0) \quad (20)\]

Plots of the magnitude and phase are shown in figure 5.

It is customary to plot magnitude in decibels, that is, \(20 \log_{10} (e_0/e_1)\) against the logarithm of frequency. Then the magnitude plots may be approximated by straight lines.

Phase is simply plotted versus the logarithm of frequency. Notice that the frequency axis is plotted as the ratio of frequency to cutoff frequency in radians per second: \(\omega/\omega_0\). Without change, the frequency axes could equally well be the ratio of frequency to cutoff frequency in Hertz, \(f/f_0\).

There is much that can be said about these plots. See for example [5].

The plotting routines for the Gnuplot program, are in section 6.3 on page 37.

3.1.2 Measurement

Figure 6 shows measurements on an RC lowpass filter with \(R = 10k\Omega\) and \(C = 100nF\). The cutoff frequency \(f_0\) is then 160Hz.

In figure 6(a), the generator was set to 160Hz. The output waveform (smaller of the two sine waves) lags the input by 45 degrees and is smaller by 3db.

Figure 6(b) shows the results of a swept-frequency measurement with the Vector Network Analyser (VNA)\(^8\) software operating on a Syscomp WGM-101 waveform generator and DSO-101 oscilloscope [11].

The magnitude and phase are as predicted by the graphs of figure 5.

\(^8\)A Vector Network Analyser is sometimes referred to as a Bode Plotter.
3.2 Transfer Function of Low Pass LR Filter

The LR lowpass filter is shown in figure 7. Now we’ll determine the frequency response of this filter. As we’ll see, this is very similar to the RC lowpass filter of the previous section.

The inductor and resistor form a voltage divider, so we can write:

\[
\frac{e_o}{e_i} = \frac{Z_2}{Z_1 + Z_2} \quad (21)
\]

where

- \(e_o\): AC Output voltage voltage from the circuit
- \(e_i\): AC Input voltage to the circuit
- \(Z_1\): Impedance of the upper half of the voltage divider (the inductor)
- \(Z_2\): Impedance of the bottom half of the voltage divider (the resistor)

Now substitute \(sL\) for \(Z_1\), \(R\) for \(Z_2\) and do some algebra:

\[
\frac{e_o}{e_i} = \frac{R}{R + sL} = \frac{1}{1 + \frac{sL}{R}} \quad (22)
\]

This time, the we’ll see that the quantity \(L/R\) is the time-constant of the network and the inverse of the lowpass cutoff frequency. Consequently, \(\tau = L/R\) for the LR lowpass.

Then we could rewrite equation 22 as:

\[
\frac{e_o}{e_i} = \frac{1}{1 + s\tau} \quad (23)
\]

A Wonderful Thing happens at this point: this equation 23 for the LR lowpass filter has exactly the same form as equation 13 on page 7, which we previously found for the RC lowpass filter. Consequently, for equal values of time constant they will have exactly the same magnitude and phase response\(^9\).

Consequently, we can simply recycle the remainder of the information for the RC lowpass filter, recognizing that the time constant is \(\tau = RC\) for the RC lowpass and \(\tau = L/R\) for the LR lowpass.

3.2.1 Measurement

For these measurements, the inductor is 30mH and the resistor 2k\(\Omega\), for a cutoff frequency \(f_o\) of 10kHz. Measurements are shown in figure 8 on page 11. Based on the results of the previous section, the measurements should be similar to those of the RC lowpass filter, adjusted for the different cutoff frequency.

The sine wave display of figure 8(a), taken at 10kHz, is essentially as expected. The output waveform lags the input by 45 degrees.

The VNA plot also shows the expected behaviour at low and mid frequencies. However, a careful examination at high frequencies shows something unexpected: the rolloff is slightly more than the expected 20db/decade. As well, the phase is headed for a larger angle than 90°. Inductance is often accompanied by significant series resistance and parallel stray capacitance (so-called parasitic components), so they may be factor.

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\(^9\)This assumes ideal behaviour of the components. In practice, the components have various forms of non-ideal behaviour, and that can be important.
3.3 Transfer Function of a Second Order RLC Lowpass Filter

Like the RC filter of figure 4, figure 9 is a lowpass filter: it passes low frequencies without attenuation and it attenuates high frequencies. However, the attenuation above the cutoff frequency occurs at twice the rate of the RC lowpass filter and there is less attenuation at the cutoff frequency.

Furthermore, the RC lowpass filter is a first order filter, because the largest power of $s$ is unity. The RLC lowpass filter is second order, since it contains an $s^2$ term.

As we did in the previous section, we’ll develop the transfer function and then massage it into a form suitable for plotting the frequency and phase response. Again, this illustrates the Laplace transform operator $s$ as shorthand notation for $j\omega$.

Analysis

Treating the network of figure 9 as a voltage divider, the gain of the network is

$$\frac{e_o}{e_i} = \frac{Z_2}{Z_1 + Z_2}$$

where $Z_1$ and $Z_2$ are the upper and lower halves of the voltage divider.

$$Z_1 = sL$$

$$Z_2 = \frac{1}{sC} || R$$
Then the transfer function is

\[
\frac{e_o}{e_i} = \frac{R}{1 + sRC}
\]

Now we will manipulate this equation into a more useful form, in which the the \(s^2\) term in the denominator has a coefficient of 1. Divide the denominator by the factor \(R\) in the numerator and then factor \(LC\) out of the denominator:

\[
\frac{e_o}{e_i} = \frac{1}{LC \left( s^2 + \frac{1}{RC}s + \frac{1}{LC} \right)}
\]

(24)

Now we can substitute for some of these quantities. The resonant frequency for an \(RLC\) circuit is

\[
\omega_o = \frac{1}{\sqrt{LC}}
\]

(25)

so we can substitute \(\omega_o^2\) for \(1/LC\) in equation 24.

We can also introduce a \(Q\) factor, which is the ratio of the resistance in the circuit to the reactance at resonance. At resonance, the inductive and capacitive reactance are equal, so we could choose either one. We’ll choose the inductive reactance:

\[
Q = \frac{R}{\omega_o L}
\]

(26)

Since at resonance the inductive and capacitive reactances are equal:

\[
\omega_o L = \frac{1}{\omega_o C}
\]

(27)

Substituting for \(\omega_o L\) in equation 26.

\[
Q = \omega_o RC
\]

(28)

and

\[
\frac{1}{RC} = \frac{\omega_o}{Q}
\]

(29)

So we can substitute \(\omega_o/Q\) for \(1/RC\) in equation 24. Then

\[
\frac{e_o}{e_i} = \frac{\omega_o^2}{s^2 + \omega_o s + \omega_o^2}
\]

(30)

This is the standard form for the 2nd order lowpass filter.

3.3.1 Magnitude and Phase of the Lowpass RLC Filter

In equation 30, move the \(\omega_o^2\) term in the numerator into the denominator.

\[
\frac{e_o}{e_i} = \frac{1}{\left( \frac{s^2}{\omega_o^2} + \frac{s}{\omega_o Q} + 1 \right)}
\]

(31)

Now replace each occurrence of \(s\) by \(j\omega\). Since \(j = \sqrt{-1}\), \(j^2 = -1\).
case of the RC filter, we’d like the result in decibels, so we take the response in the two decades around the cutoff frequency. We could plot the function over the range of frequency to cutoff frequency. We could plot the function over the range $-50$ to $10$ Amplitude, db

Then we have

For notational convenience replace $\omega/\omega_c$ with the $x$ axis variable, which I’ll call $x$. Then $x$ represents the ratio of frequency to cutoff frequency. We could plot the function over the range $x = 0.1$ to $x = 10$ to get an idea of the response in the two decades around the cutoff frequency.

Then we have

Collecting real and imaginary terms, we have

Magnitude

The magnitude is equal to the square root of the sum of the real and imaginary components each squared. As in the case of the RC filter, we’d like the result in decibels, so we take $20 \log_{10}$ of the result:

Now, if you consider that

$$
\log_{10} \sqrt{\frac{1}{A}} = \log_{10} \left( A^{-\frac{1}{2}} \right) = -\frac{1}{2} \log_{10}(A)
$$
Then we can write equation 34 as:

$$\frac{e_o}{e_i} = -10 \log_{10} \left[ (1 - x^2)^2 + \left( \frac{x}{Q} \right)^2 \right] \quad (36)$$

Equation 36 is in a form that can be plotted, as shown in figure 10(a).

**Phase**

If a complex number $a$ is written in rectangular format as for example $a = x + jy$, the phase angle is given by

$$\angle a = \tan^{-1} \left( \frac{y}{x} \right) \quad (37)$$

where $x$ is the so-called real part and $y$ is the imaginary part. A complex number may also be written in polar format as for example $a = R \angle \alpha$ where $R$ is the magnitude and $\alpha$ is the phase angle.

Then it can be shown (see section 6.2 on page 35) that

$$\frac{1}{x + jy} = -\tan^{-1} \left( \frac{y}{x} \right) \quad (38)$$

Applying this concept to equation 33, we have that

$$\angle \frac{e_o}{e_i} = -\tan^{-1} \left( \frac{x/Q}{1 - x^2} \right) \quad (39)$$

Equation 39 is plotted in figure 10(b).

Some points of interest:

- The phase changes from $0^\circ$ to $-180^\circ$, passing through $-90^\circ$ at the resonant frequency $\omega_o$.
- The rate of change of phase in the vicinity of $\omega_o$ increases with increasing $Q$ factor (which corresponds to decreasing values of damping $\delta$).

### 3.3.2 Measurement

The experimental lowpass filter was constructed according to figure 9 on page 11 with $L = 30\, \text{mH}$, $C = 1\, \mu\text{F}$, $R = 620\, \Omega$\textsuperscript{10}.

Then the cutoff frequency $f_o$ for the filter (equation 25) is:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

Since $\omega_o = 2\pi f_o$, then

$$f_o = \frac{1}{2\pi\sqrt{LC}} = 919 \, \text{Hz}$$

The $Q$ factor (equation 26) is:

$$Q = \frac{R}{\omega_o L} = \frac{620}{2\pi \times 919 \times (30 \times 10^{-3})} = 3.58$$

\textsuperscript{10}The waveform generator has an internal resistance of $75\, \Omega$ that should be taken into account. However, it turns out this resistance is small compared to the circuit impedances and does not have a major effect on the circuit operation so for our purposes can be ignored. If there is any doubt in the matter, its effect must be investigated.

As well, the inductor has some DC resistance and the capacitor some ESR (equivalent series resistance) that should be taken into account in a completely precise analysis. These also have been ignored in this measurement.

Including these components in the analysis complicates it to the point that a circuit simulation is probably the most efficient approach.

In a teaching environment such as an EE lab, it’s up to the designer of the measurement exercise to ensure that these so-called parasitic components do not obscure the behaviour that is being illustrated.
We expect to see a lowpass filter that rolls off above 919Hz with a significant peak in the region of the cutoff frequency.

The measurement results are shown in figure 11. Figure 11(a) shows the waveforms below cutoff frequency. As expected, they are in phase and nearly equal in magnitude.

Figure 11(b) shows the waveforms at $f_o$, the cutoff frequency. The waveforms are 90° out of phase and the output is larger than the input, due to peaking at this frequency.

Figure 11(c) shows the waveforms above cutoff. The output is now 180° out of phase with the input and much reduced in magnitude.

Figure 11(d) shows the network analyser measurement. As expected, there is peaking around $f_o$ and the output drops at 40db/decade above the cutoff frequency. The phase display can only accommodate a range of $+90°$ to $-90°$, so we see an abrupt change in phase at the resonant frequency. At high frequencies, the phase shift is $180°$ as expected.
3.4 Transfer Function of the Two Stage (Second Order) RC Lowpass Filter

The two-stage RC lowpass filter is shown in figure 12(a) and a rearrangement for analysis in figure 12(b).

We can treat this network as a two-stage voltage divider. Working backward from output to input:

- The output voltage \( e_o \) is divided down from the intermediate voltage \( e_a \) by resistor \( R_2 \) and capacitor \( C_2 \).
- The intermediate voltage \( e_a \) is divided down from \( e_i \) by a voltage divider consisting of resistor \( R_1 \) and the network \( Z \). The network \( Z \) is composed of capacitor \( C_1 \) in parallel with the series combination of \( R_2 \) and \( C_2 \).

After much algebraic crank-turning, we obtain:

\[
\frac{e_o}{e_i} = \frac{1}{s^2 + \left(\frac{R_1C_2}{R_1R_2C_1C_2} + \frac{1}{R_1R_2C_1C_2}\right)s + \frac{1}{R_1R_2C_1C_2}} \times \frac{1}{R_1R_2C_1C_2}
\]  

On most occasions, the resistors and capacitors are equal, so let us make:

\[  R_1 = R_2 = R, \quad C_1 = C_2 = C \]

Many things cancel at this point, and after some cosmetic surgery we have as a result:

\[
\frac{e_o}{e_i} = \frac{1}{s^2 + \frac{3}{RC}s + \frac{1}{R^2C^2}}
\]  

Equation 41 is equivalent to the standard form of equation 42 if we make:

\[
\frac{\omega_o}{\sqrt{3}} = \frac{1}{Q}, \quad Q = \frac{1}{3}
\]

Then the magnitude and phase are as shown on the plots for the standard form, figure 10 on page 13. In the case of the two stage RC lowpass, the value of \( Q \) is 1/3 which is one of the traces in figure 10.

Now let’s think about what this means. Ideally, we’d like our filter to have no attenuation in the passband (below the cutoff frequency) and much attenuation in the stop band (above the cutoff frequency).

According to figure 10 that ideal is achieved most closely for the second-order lowpass filter when the the Q value is about 1. The response drops at 40db/decade above the corner frequency and the attenuation at the cutoff frequency is approximately zero.

In contrast, the dual-section RC filter makes the transition very gradually between passband and stop band. It attenuates the signal by 10 db at the cutoff frequency. It is only at ten times the cutoff frequency that the attenuation rate approaches 40db/decade. So the dual section RC filter is not a great performer in the frequency domain.
4 Time Domain Response

In section 3 we determined the frequency response behaviour of these four lowpass filters. Conceptually, we would apply to the input of the filter a sine wave of various arbitrary frequencies and then measure the relationship between the input and output sine waves at these frequencies.

The substitutions we used are useful in determining the frequency response of networks. That work requires a fair amount of algebra, but nothing new relating to the Laplace transform.

Now we will consider what happens when we apply various signals that are arbitrary in the time domain: that is, they have arbitrary shapes. The Laplace transform is particularly useful for this type of analysis, when it is necessary to determine the time-response of a particular electrical circuit.


2. Determine the Laplace transform of the input signal. Some input signals are simple enough that the transform is known. For example, the Laplace transform of the unit impulse is simply 1. The Laplace transform of the unit step is $1/s$. In other cases, a more complex signal is specified in the problem and some work is required to determine the transform.

3. Determine the Laplace transform of the network transfer function as we showed in section 3.1 and section 3.3.1. This can usually be written out by inspection of the circuit.

4. Multiply the input signal of step 2 by the transfer function of step 3 to obtain the Laplace transform of the output signal.

5. Take the inverse Laplace transform of the output signal to obtain the output voltage as a function of time.

Next, we introduce two useful tools for this type of analysis: a computer algebra program (Maxima) (section 4.1) and a table of Laplace transforms (section 4.2).

In sections 4.3, 4.5 and 4.5 we use these tools to develop the Laplace transforms of the impulse, step and ramp input signals, or forcing functions as they are known.

Then we apply each of these signals to the RC lowpass network and determine the resultant output voltage-time signal.

4.1 Computer Algebra: Using Maxima

A computer algebra system (CAS) [2] manipulates the symbols of math equations according to the rules of algebra, calculus and other branches of mathematics. A computer algebra system can remove much of the labour from Laplace and Inverse Laplace transforms. In this paper we use the open-source CAS Maxima program (figure 13).\footnote{Maxima is roughly comparable to the proprietary programs Mathematica and Maple. Notice that Matlab and open-source equivalent Octave (widely used in universities for simulation of electronic systems) is (to quote Wikipedia): a numerical computing environment and programming language. In other words, Maxima, Mathematica and Maple work with symbols. Matlab and Octave work with numbers. For a general-purpose solution, not tied to specific values in the circuit, you need a program that works with symbols.}

A list of alternative CAS programs is in [3].

Apart from a few introductory examples given below, we show only commands that are relevant to the business at hand. A complete manual for Maxima (860 pages!) is available at the Maxima home page [4].

Here are some examples of Maxima at work:

Big Numbers

The (%i28) and (%o28) are Maxima input and output prompts.

(%i28) 12^26;
(%o28) 11447545997288281555215581184
Solving an Equation

First, we define the equation

\( a \times x^2 + b \times x + c; \)

Then we ask Maxima to solve it. The \( \% \) symbol means ‘the previous equation’.

\( \text{(s24) solve (, x);} \)

\( \text{(o25) [} \times = \frac{\sqrt{b^2-4ac} - b}{2a} , x = \frac{\sqrt{b^2-4ac} + b}{2a} \text{]} } \)
Differentiating

(%i30) diff (cos(x), x);
(%o30) - sin(x)

(%i32) diff ((sin(x))**3, x);
(%o32) 3 cos(x) sin(x)**2

Integrating

(%i31) integrate(-sin(x), x);
(%o31) cos(x)

Complex Math

(%i is the complex operator, also known as j.)
(%i33) rectform(5.0 / (3.0+2.0*%i) + 4.0 / (8.0*%i+5.0));
(%o33) 1.378565254969749 - 1.128781331028522 i

4.2 Table of Transforms

<table>
<thead>
<tr>
<th>Description</th>
<th>Time Domain Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Impulse</td>
<td>δ(t)</td>
<td>1</td>
</tr>
<tr>
<td>2 Unit Step</td>
<td>u(t)</td>
<td>1/s</td>
</tr>
<tr>
<td>3 Unit Ramp</td>
<td>r(t)</td>
<td>1/s^2</td>
</tr>
<tr>
<td>4 Integration</td>
<td>∫ f(t)</td>
<td>1/s F(s)</td>
</tr>
<tr>
<td>5 Differentiation</td>
<td>d/dt f(t)</td>
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<tr>
<td>6 Linearity</td>
<td>k f(t)</td>
<td>kF(s)</td>
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<tr>
<td>7 Exponential</td>
<td>exp^αt</td>
<td>1/s - α</td>
</tr>
<tr>
<td>8 Superposition</td>
<td>A f(t) + B g(t)</td>
<td>AF(s) + BG(s)</td>
</tr>
</tbody>
</table>

4.3 Forcing Function: Unit Impulse

The simplest forcing function is the unit impulse δ(t). The theoretical abstraction of the unit impulse is a pulse with area unity, zero width, and infinite amplitude.12

One way to think of the unit impulse is shown in figure 14(a). A unit impulse is the limiting case of a pulse that is reduced in duration while keeping the area unchanged.

Another view is shown in figure 14(b). The unit impulse is the slope (differential d/dt) of the rising portion of a unit step waveform. As the slope increases, the width of the unit impulse decreases while its amplitude increases to maintain unity area. In the limit, as the rising portion of the step approaches vertical, the unit impulse approaches the theoretical abstraction of zero width, infinite height, unity area.

When is the unit impulse useful? The Laplace Transform of the unit impulse is simply 1. As a result, if you apply a unit impulse to a system, then the output is the system transfer function. This makes it very simple to determine the

---

12See footnote on page 3 for explanation of the terms magnitude and amplitude.
For example, to measure the transfer function of a mechanical system apply an impulse by hitting it with a hammer. Record the output. Now, this impulse is probably not ideal: it has finite duration and finite amplitude. However, if the pulse is short compared to any of the time constants in the system, then it will appear as an impulse.

With the time-domain response to an impulse input you can have enough information to determine the response of a system to any input signal,

1. Apply an impulse to the system and record the corresponding output. This is the system impulse response in the time domain.

2. Use the Laplace transform to convert this time-domain system impulse response to the Laplace domain.

3. Choose an input signal and obtain its Laplace transform. Multiply the Laplace domain impulse response by the Laplace domain input signal. This yields the Laplace domain output signal.

4. Take the inverse Laplace transform of this output signal to determine the output signal in the time domain.

The impulse technique cannot be used in some practical applications. Perhaps the system cannot be disturbed with an impulse. Or an impulse drives the system into non-linear behaviour. Or it may be that there is a signal-noise problem: the output signal is too weak at the maximum allowable impulse signal. In those cases, cross-correlation may be more useful. Low-level random noise is fed into the system. The output signal is cross-correlated against the input. The resulting correlation function is the impulse response of the system.

The amplitude of an impulse may be scaled by some factor $K$. Then the Laplace transform of the scaled impulse is $K \times 1 = K$.

It is a bit tricky to prove that the Laplace transform of the unit impulse is 1. Maxima is not much help. A proof is shown in section 6.4 on page 39.
4.4 Forcing Function: Unit Step

The unit step is shown in figure 15. A step waveform of height 1 unit is the time integral of a unit impulse. In the Laplace domain, integration is accomplished by multiplying by $1/s$. Consequently, the Laplace transform of the unit step is $1/s$ times unity, or $1/s$.

We can obtain the same result using the definition of the Laplace transform, as shown in section 6.5 on page 40.

Or we can use Maxima to do this. From the Maxima manual:

\[
\textit{laplace}(expr, t, s) \text{ attempts to compute the Laplace transform of } expr \text{ with respect to the variable } t \text{ and transform variable } s.
\]

In this case, a unit step, \( expr \) is simply 1.

\[
\begin{align*}
(C22) \ & \text{laplace (1, t, s);} \\
(D22) \ & \frac{1}{s}
\end{align*}
\]

This also works in reverse. Again, from the manual:

\[
\textit{ilt}(expr, s, t) \text{ computes the inverse Laplace transform of } expr \text{ with respect to the } t \text{ and variable } s. \text{ expr must be a ratio of polynomials whose denominator has only linear and quadratic factors.}
\]

This time, \( expr \) is $1/s$ and the inverse transform yields a unit step waveform.

\[
\begin{align*}
(C21) \ & \text{ilt (1/s, s, t);} \\
(D21) \ & t
\end{align*}
\]

In practice, waveforms are rarely one unit in amplitude, in which case the amplitude of the step may be scaled by some factor $K$.

4.5 Forcing Function: Unit Ramp

The unit ramp is shown in figure 16, a ramp waveform of 1 unit increase per unit of time. As the step waveform is an integral of the impulse, so is the ramp an integral of the step. Again, in the Laplace domain integration is accomplished by multiplying by $1/s$. Consequently, the Laplace transform of the unit ramp is $1/s^2$.

This is confirmed by Maxima, for the transform and inverse transform:

\[
\begin{align*}
(C15) \ & \text{laplace(t, t, s);} \\
(D15) \ & \frac{1}{2s} \\
(C14) \ & \text{ilt (1/s^2, s, t);} \\
(D14) \ & t
\end{align*}
\]
As in the case of the impulse and step waveforms, the magnitude of the ramp may be scaled by some factor $K$.

### 4.6 Unit Impulse and RC Lowpass

Now we are in a position to analyse some simple circuits. Our first case is to determine the output signal when the RC lowpass filter is driven by a unit impulse.

The Laplace transform of the transfer function of the RC lowpass filter was derived previously as equation 13 on page 7:

$$\frac{e_o}{e_i} = \frac{1}{1 + s\tau}$$  \hspace{1cm} (43)

The Laplace transform of the output signal is the product of the input signal (unity) and the transfer function, equation 43. To find the output signal as a function of time, we need the inverse transform for equation 43. There are two ways to do this: with the computer algebra system (Maxima) and manually, with a table of transforms.

#### 4.6.1 Inverse Transform, Using Maxima

Maxima can find the inverse transform:

(C18) \text{ilt}(a/(s+a), s, t);

(D18) \quad -a t \quad \text{%E}

where %E is the constant $e$. Back substitute $a = 1/\tau$ and we obtain:

$$\frac{e_o(t)}{e_i(t)} = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$  \hspace{1cm} (44)

#### 4.6.2 Inverse Transform, Using Tables

Consulting the table of Laplace transforms (section 4.2 on page 19), entry 7 looks as if it might be useful.

$$\frac{1}{s - a} \leftrightarrow e^{at}$$  \hspace{1cm} (45)

We’ll manipulate equation 43 into that form.

$$\frac{e_o(s)}{e_i(s)} = \frac{1}{1 + s\tau} \quad \Rightarrow \quad \frac{1}{\tau} \left( \frac{1}{s + \frac{1}{\tau}} \right)$$  \hspace{1cm} (46)

Then $-1/\tau$ in equation 46 is equivalent to $a$ in the transform pair of equation 45. We also have a leading constant $1/\tau$ in equation 46 to take into account. According to entry 6 of the table, this remains unchanged from the Laplace to the time domain. Then the inverse transform of equation 46 is:
\[
\frac{e_o(t)}{e_i(t)} = \frac{1}{e} - \frac{t}{\tau}
\]

which matches equation 44.

\[e(t) = Ke^{\frac{-t}{\tau}}\]

\[\tau = RC\]

\[V_{\text{init}} = \frac{K}{\tau}\]

\[0.368V_{\text{init}}\]

\[t = 0 \quad t = RC\]

\[v(t)\]

(a) Theoretical Waveform

(b) DSO-101 Oscilloscope Display

(c) Display Magnified

(d) WGM-101 Generator Settings

Figure 18: Impulse and RC Lowpass

4.6.3 Measurements

Figure 18(a) shows the theoretical result of an impulse measurement on an RC lowpass network. The initial magnitude of the output waveform is equal to \(K/\tau\), where \(K\) is the area of the impulse in volt-seconds and \(\tau\) is \(RC\), the time constant of the network. After the initial output, the output decays to \(1/e\) of its original value in one time constant.

Figure 18(b) shows an impulse measurement on an RC lowpass network with \(R = 9880\Omega\) (10k\(\Omega\) nominal), \(C=95.6nF\) (100n\(F\) nominal). Then the time constant for this network is \(\tau = RC = 9880 \times (95.6 \times 10^{-9}) = 0.944\) milliseconds.
We must check the source and load impedance of the measuring equipment to determine that it does not affect the measurement. The output impedance of the generator is 75Ω, much less than the resistance, so it may be neglected. The input impedance of the oscilloscope is 1MΩ || 20pF, again large enough to be neglected.

In order that the input pulse be considered an impulse, the pulse width must be much less than the time constant of the network and there must be sufficient time for the output transient to die away. In this case, the generator is set to a duty cycle of 1% which yields a pulse of width 50μSec, one tenth of the time constant. The repetition rate is 200Hz so the time between pulses is 5 msec, 5 time constants. The pulse is unipolar so the generator output is offset by +4 volts with a pulse amplitude of 8 volts peak.

Figure 18(d) shows the control setup for the generator.

Results

Figure 18(b) shows the measured input impulse (upper trace) and output waveform (lower trace).

The measurement cursors and adjustable trigger point of the DSO-101 oscilloscope make it very convenient to measure the voltage and time values of the output waveform.

The input area of the pulse is 8 volts times the pulse width of 50μSec for a total of 400×10^{-6} volt-seconds. This is the value of $K$ in the equation of figure 18(a). Then the peak value of the output pulse is theoretically $V_{init} = K/\tau = 400$mV. The measured value is 465mV.

The output waveform should decay to $1/e = 0.368$ of its original value in one time constant, 0.944mSec in this case. In fact, a decay from 465mV to 173mV occurs in 870μSec.

Figure 18(c) shows an expanded view of the pulse interval. Notice how the capacitor voltage ramps up during the pulse. During the charging interval the waveform is exponential but because the capacitor voltage is always much less than the pulse voltage, the voltage across the resistor is approximately constant and the capacitor charging current is approximately constant. As a result, the capacitor charging waveform is approximately linear. This is a useful approximation to keep in mind for some circuit analysis problems.

4.7 Unit Step and RC Lowpass

Now we’ll determine the unit step response of the RC low-pass filter, figure 19.

Again, we start with the Laplace transform of the transfer function of the RC lowpass filter, equation 13:

$$\frac{e_o}{e_i} = \frac{1}{1 + s\tau} \quad (48)$$

Again, it looks as if the transform

$$\frac{1}{s - a} \leftrightarrow e^{at} \quad (49)$$

might be useful. Rearrange equation 48 and put $1/\tau = a$ to simplify the notation:

$$\frac{e_o}{e_i} = \frac{a}{a + s} \quad (50)$$

The output signal as a function of time is the inverse Laplace transform of the product of the unit step input signal ($1/s$) and the transfer function, equation 50.

$$\frac{e_o(t)}{e_i(t)} = \mathcal{L}^{-1} \frac{1}{s} \left( \frac{a}{a + s} \right) \quad (51)$$

Now we must find this inverse transform. We’ll show two methods, a computer algebra method using Maxima and a manual method.
4.7.1 Method 1: Using Maxima

Equation 51 is of the form

\[ \frac{a}{s} \left( \frac{1}{s + a} \right) \]

where \( a = 1/\tau \). Here is Maxima finding the inverse transform:

\[
\text{(C25) ilt( (a/s)*(1/(s+a)), s, t);} \\
\text{(D25)} \\
1 - \%e^{-a \cdot t}
\]

Back substitute \( a = 1/\tau \) and the result is:

\[
\frac{e_o(t)}{e_i(t)} = 1 - e^{-t/\tau}
\]  \hspace{1cm} (52)

This equation is the time response of an RC lowpass network to a unity-step input. If the step is \( K \) volts in magnitude, then the response is multiplied by \( K \).

4.7.2 Method 2: Using Transform Table Entry

Here is the manual method. It’s a lot more work than using Maxima, but nothing is hidden ‘behind the curtain’.

We begin with equation 51. We need to apply partial fraction expansion to get equation 51 into a suitable form for the transforms in the table (section 4.2). Maxima can do partial fraction expansions, but we’ll do it manually. In equation 54 we reorganize equation 51 into the sum of two terms. Each denominator is one that has a recognizeable entry in the table of Laplace transforms. Now we need to determine the numerators \( x \) and \( y \) to satisfy this form.

It’s also a good idea to move the numerator constant \( a \) out of the partial fraction expansion (as we’ll see later), so we’ll do that now.

\[
\frac{1}{s} \left( \frac{a}{a+s} \right) = \frac{1}{s} \left( \frac{1}{a+s} \right)
= \frac{a \left[ x + \frac{y}{s} \right]}{s(a+s)}
= \frac{a \left[ xa + xs + ys \right]}{s(a+s)}
= \frac{a \left[ xa + s(x+y) \right]}{s(a+s)}
\]

Comparing the numerator of equation 54 with the numerator of equation 51, we can write the following:

\[
x = \frac{1}{a}
\]

Also, since the \( s \) term in the numerator in the numerator of equation 51 is non-existent:

\[
x + y = 0 \hspace{1cm} y = -x
= \frac{1}{a}
\]

Check these values by back-substituting for \( x \) and \( y \) in equation 53:
\[
\frac{1}{s} \frac{a}{a+s} = a \left[ \frac{x}{s} + \frac{y}{a+s} \right] \\
= a \left[ \frac{1}{as} - \frac{1}{a(a+s)} \right] \\
= \frac{a}{s(a+s)}
\] (57)

This confirms that \(x\) and \(y\) are correct. Now we can proceed with the inverse transform.

\[
\frac{e_0(t)}{e_i(t)} = \mathcal{L}^{-1} \left( \frac{a}{a+s} \right) \frac{1}{s} \\
= \mathcal{L}^{-1} a \left( \frac{1}{as} - \frac{1}{a(a+s)} \right) \\
= \mathcal{L}^{-1} \left( \frac{1}{s} - \frac{1}{a+s} \right)
\] (58)

Consulting the table of Laplace transforms (section 4.2 on page 19), entry 8 indicates that each of these terms can be treated separately. According to entry 2 of the table, the \(1/s\) term transforms to \(e^{-at}\). The \(1\) term in the numerator is a constant so it transforms unchanged. The \(1/(a+s)\) term transforms to \(e^{-at}\). Putting this together and back substituting \(1/\tau = a\), we have:

\[
\frac{e_0(t)}{e_i(t)} = \mathcal{L}^{-1} \left( \frac{1}{s} - \frac{1}{a+s} \right) \\
= +1 - 1e^{-at} \\
= 1 - e^{-t/\tau}
\] (59)

which is the same result that Maxima found for us in equation 52.

4.7.3 Sanity Check

It’s always a good idea to check the reasonableness of an equation against the physical behaviour of the circuit.

When the unity-value step is applied to the RC lowpass circuit, the capacitor will begin to charge and that process will continue until the capacitor voltage is equal to the input voltage. Let’s examine equation 59 to see if that is true.

At time \(t = 0\), the exponential term \(e^{-t/\tau}\) is then \(e^0 = 1\), so the output voltage is zero.

At time \(t = [\text{Very Large Value Compared to } \tau]\), then \(e^{-[\text{Large Value}] \approx 0}\) and the output voltage is equal to the input voltage.

Both of these agree with the physical situation.

We could also put \(t = \tau\) in which case the output voltage is \(1 - e^{-1} = 0.633\). That is, the charging is 63% complete after one time constant.

4.7.4 Measurements

Figure 20(a) shows the the theoretical step response of an RC lowpass network. The output voltage rises up to the input voltage at a rate defined by the time constant. Figure 20(b) shows the measured response of an RC lowpass network to a 5 volt input step, where \(R = 2.075\Omega\) and \(C = 1\mu\text{F}\). The time constant is then \(RC = \tau = 2.075\text{msec}\). At 63% of the final voltage (3.15 volts) the time elapsed is about 2msec, as predicted.
4.8 Unit Ramp and RC Lowpass

Now we’ll do the unit ramp response of the RC lowpass filter. Again, we start with the Laplace transform of the transfer function of the RC lowpass filter, equation 13:

\[
\frac{e_o}{e_i} = \frac{1}{1 + s\tau} \quad (60)
\]

Again, it looks as if the transform might be useful. Rearrange equation 48 and put \(1/\tau = a\) to simplify the notation:

\[
\frac{e_o}{e_i} = \frac{a}{a + s} \quad (62)
\]

The Laplace transform of the input ramp signal is \(1/s^2\). The output signal as a function of time is the inverse Laplace transform of the product of the input signal and the transfer function, equation 63.

\[
\frac{e_o(t)}{e_i(t)} = \mathcal{L}^{-1} \left( \frac{a}{a + s} \right) \quad (63)
\]

It simplifies things to move the \(a\) term in the numerator out in front of the expression as a constant.

\[
\frac{e_o(t)}{e_i(t)} = \mathcal{L}^{-1} a \left[ \frac{1}{s^2} \left( \frac{1}{a + s} \right) \right] \quad (64)
\]
4.8.1 Inverse Transform, Using Maxima

Maxima can find the inverse transform:

\[(C26) \text{ilt} \left( \frac{a}{(s^2)} \ast \frac{1}{(s+a)}, s, t \right) = \frac{a \cdot t}{a} - \frac{1}{a} \]

\[(D26) \quad \frac{-a \cdot t}{a} + \frac{1}{a} \]

Back substitute \(a = 1/\tau\) and rearrange, then we have:

\[
\frac{e_o(t)}{e_i(t)} = t - \tau \left(1 - e^{-t/\tau}\right) \quad (65)
\]

This equation is the time response of an RC low-pass network to a unity-ramp input. If the ramp is \(K\) volts/second in magnitude, then the response is multiplied by \(K\) (figure 22).

Let’s think about this equation. It says that the ramp input generates a ramp output minus something. Looking at the something part, the exponential term disappears as time becomes large. In other words, everything after the ramp eventually becomes a constant equal to \(\tau\). So the output tracks the input with a constant difference (control systems people would call this an error) of \(\tau = RC\).

4.8.2 Inverse Transform, Using Algebra and Transform Table

We can obtain the same result by hand. Start with equation 64 above.

If we extract a \(1/s\) term from the right side, then what remains inside the square brackets is something we’ve already done:

\[
\frac{e_o(t)}{e_i(t)} = L^{-1} \frac{a}{s} \left[ 1 \frac{1}{s + a} \right] \quad (66)
\]

We previously showed (equation 58) that the expression inside the square brackets can be expanded in fractions as follows:

\[
\frac{1}{s} \left( \frac{1}{a + s} \right) = \frac{1}{as} - \frac{1}{a(a + s)} \quad (67)
\]

Back substitute from equation 67 into equation 66 and we have:

\[
\frac{e_o(t)}{e_i(t)} = L^{-1} \frac{a}{s} \left[ \frac{1}{as} - \frac{1}{a(a + s)} \right] = L^{-1} \frac{1}{s} \left[ \frac{1}{as} - \frac{1}{s(a + s)} \right] = L^{-1} \frac{1}{s^2} - \frac{1}{s(a + s)} \quad (68)
\]

We can do the same substitution on the second term in the equation, again from equation 58:
\[
e_{o}(t) = \mathcal{L}^{-1} \left( \frac{1}{s^2} - \frac{1}{s(a+s)} \right)
= \mathcal{L}^{-1} \left( \frac{1}{s^2} - \frac{1}{as} + \frac{1}{a(a+s)} \right)
= \mathcal{L}^{-1} \left( \frac{1}{s^2} - \frac{1}{as} + \frac{1}{a(a+s)} \right)
\]

Now we can proceed with the inverse transform.
Consulting the table of Laplace transforms (section 4.2 on page 19), entry 8 indicates that each of these terms can be treated separately.

\[
\mathcal{L}^{-1} \frac{1}{s^2} = t
\]
This is the ramp signal.

\[
\mathcal{L}^{-1} \frac{1}{as} = \frac{1}{a}
\]

\[
\mathcal{L}^{-1} \left( \frac{1}{a(s+a)} \right) = \frac{1}{a} e^{-at}
\]

Putting this together and back substituting \(1/\tau = a\), we have:

\[
\frac{e_{o}(t)}{e_{i}(t)} = \mathcal{L}^{-1} \left( \frac{1}{s^2} - \frac{1}{as} + \frac{1}{a(a+s)} \right)
= t - \frac{1}{a} + \frac{1}{a} e^{-at}
= t - \tau \left( 1 - e^{-t/\tau} \right)
\]

This matches the result obtained using Maxima.

4.8.3 Measurement

Figure 23(a) shows the theoretical ramp response of an RC lowpass network. This measurement requires a waveform that increases during a ramp interval of time and then drops back to zero during a reset interval. Triangle and sawtooth waveforms commonly available from function generators provide a ramp but not the necessary reset interval.

In this case, the necessary ramp-reset waveform was created with equal time assigned for the ramp and the reset. The Wavemaker utility program was used to draw the waveform\textsuperscript{13}. Then this waveform was loaded into the WGM-101 waveform generator. The DSO-101 can then reproduce this arbitrary waveform at the desired frequency and amplitude.

The output voltage rises at the same rate as the input, but delayed by one time constant. Figure 23(b) shows the measured response when the time constant is \(RC = \tau = 945\) msec. The measured delay is 937msec.

\textsuperscript{13}Wavemaker is available as a free download from the Syscomp web page, in the Downloads section.
4.9 Two Stage (Second Order) Lowpass, Impulse Response

In section 3.4 we showed the frequency domain analysis of the two stage RC lowpass filter of figure 24.

We found that the transfer function is a second-order lowpass filter with a Q factor of 1/3. Now, in accordance with our original scenario (section 1.2), let us investigate the effectiveness of this filter in attenuating a noise spike.

We can approximate a noise spike by an ideal impulse, so we need to know the impulse response of the filter. The impulse response is the inverse Laplace transform of the transfer function. Here is how we use Maxima to determine that:

1. We enter the transfer function of the second-order RC lowpass (equation 42). In this equation, $\omega$ represents $\omega_0$:

   \[
   \frac{w^2}{(s^2 + \left(\frac{w}{q}\right)s + w^2)};
   \]

2. Maxima formats and echoes this back:

   \[
   \text{(%o48)} \quad \frac{w^2}{s^2 + \frac{w}{q} + s^2};
   \]

3. We enter the command to find the inverse Laplace transform of the previous equation:

   \[
   \text{(%i49)} \quad \text{ilt(%, s, t)};
   \]

4. Maxima asks in effect whether the transfer function is overdamped, critically damped or underdamped. If the function is underdamped, the impulse response will be oscillatory, that is, there will be overshoot and ringing.

   Is $(2q - 1) (2q + 1) w^2$ positive, negative, or zero?

In this case, the $Q$ factor is 1/3, so the correct answer is negative.
negative;

5. Maxima now determines the inverse Laplace transform.

$$f(t) = \frac{t_w e^{-q w t \sqrt{1 - 4 q^2 t_w}}}{2 q w e^{2 q w t \sinh \left(\frac{\sqrt{1 - 4 q^2 t_w}}{2 q}\right)}} \sqrt{1 - 4 q^2}$$

This is the time-domain response to an impulse.

6. We can simplify this equation somewhat before plotting:
   - Replace $w(= \omega_o)$ with $1/\tau$
   - Substitute $Q(= q) = 1/3$ in the expression

$$\frac{1}{2Q} \sqrt{1 - 4Q^2} = 1.13$$

Then we obtain the much more manageable:

$$f(t) = \frac{1}{\tau} \left[ \frac{1}{1.13} e^{-1.52t \frac{1}{\tau}} \sinh \left(\frac{1.13t}{\tau}\right) \right] \quad (74)$$

7. Replace $t/\tau$ by the variable $x$, so the horizontal axis is units of time relative to the time-constant.

8. Plot the amplitude vs time with Gnuplot or some equivalent. The result of plotting the quantity within the square brackets of equation 74 is shown as the solid-line trace in figure 25.

As in previous cases, this assumes an impulse signal that has an area of 1 volt-second. If that’s not the case, then the output must be scaled by $K$, where $K$ is the area of the impulse in volt-seconds.
4.10 Comparing the Filters

Now we are in a position to compare the single-stage (first-order) RC lowpass with the two-stage (second-order) RC filter. We’ll assume a unit impulse input signal.

The response of a single stage RC lowpass filter was shown in figure 18(a) on page 23. Assume a unit impulse ($K = 1$) and regroup the equation for comparison purposes:

$$f(t) = \frac{1}{\tau} \left[ e^{-\frac{t}{\tau}} \right]$$

(75)

As we did with equation 74 we can put $t/\tau = x$ and plot everything inside the square braces. This is shown as the dashed trace in figure 25.

Two things are immediately evident:

- the second-order RC lowpass is much more effective (by a factor of four, approximately) than the first-order RC lowpass in reducing the amplitude of a noise pulse.
- the leading edge of the second-order RC lowpass pulse rises at a much lower rate. This could be significant if the filter is being used to reduce electrical noise, which is often related to the rate of change of signals.

It should be emphasised that this is an analysis of ideal components and the non-ideal behaviour of capacitors in particular should be considered in the design of an effective noise filter. For example, the equivalent series resistance of a capacitor and its lead inductance may impact its effectiveness in this application.

As a practical consideration, the two series resistors must be proportioned so they do not cause excessive voltage drop for the DC current.

5 Arbitrary Waveforms

To this point, we have assumed that the input signal is an impulse, step or ramp function of time. Not infrequently one would like to determine the response of an electrical circuit to some arbitrary input waveform. Then it is necessary to obtain the Laplace transform for that input waveform. In this section we show various examples of such waveforms. We will need two tools: the linearity property and the time-shifting property.

5.1 Linearity Property

This is straightforward. The transform of the sum of two time functions is equal to the sum of the individual transforms.

$$\mathcal{L} \{ f_1(t) + f_2(t) \} = f_1(s) + f_2(s)$$

Then to determine the Laplace transform of some time function:

- break it up into a sum of simpler components
- determine the transform of each of these components
- add the transformed components

More generally, these components can be scaled by constants, that is:

$$\mathcal{L} \{ a f_1(t) + b f_2(t) \} = a f_1(s) + b f_2(s)$$

where $a$ and $b$ are constants.
5.2 Time Shift Property

The time shift of a time function \( f(t) \) by an amount \( a \) in the positive time direction is equivalent to multiplication of the Laplace transform \( f(s) \) by \( e^{-as} \).

The example of figure 26 shows the Laplace transform of a step function that has been time-shifted by an amount \( a \).

5.3 Example: Square Pulse

Figure 27 shows how a pulse waveform may be constructed by adding together two step functions. A positive unity step initiates the waveform. After some time \( a \) an equal negative step brings the waveform back to zero. The time domain function is:

\[
f(t) = u(t) - u(t - a)
\]  

(76)

Now we can find the Laplace transform of each of these step waveforms. From section 4.4 the transform of the first step function is:

\[
\mathcal{L}\{u(t)\} = \frac{1}{s}
\]  

(77)

The second step waveform is multiplied by a constant \(-1\) and shifted by an amount \(a\), so the Laplace transform is:

\[
\mathcal{L}\{-u(t - a)\} = -\left[\frac{e^{-as}}{s}\right]
\]  

(78)

Putting these two together, we have

\[
\mathcal{L}\{u(t) - u(t - a)\} = \frac{1 - e^{-as}}{s}
\]  

(79)

---

\(^{14}\)Figure 25 shows an infinite rate-of-rise of the voltage. In practice, the rate of rise is determined by the amplitude of the pulse and the RC values, as shown in figure 18(c) on page 23.
5.4 Example: Triangular Pulse

A triangular pulse is shown in the upper half of figure 28. Davis [12] suggests the following clever technique for determining the Laplace transform of this waveform.

Recall that a ramp waveform is the integral of a step waveform, which is in turn the integral of an impulse. Then a ramp is the double integral of an impulse. In the Laplace domain, integration is accomplished with a multiplication by $1/s$. So the upper waveform in figure 28 may be considered the double integral of the lower waveform, a series of impulses.

The first unit impulse starts a unit ramp which runs for 1 second. The second impulse, amplitude -2 units, cancels the first ramp and creates a negative-going unit ramp. That runs for a further second until it is cancelled by the third impulse, which creates a positive unit ramp.

Impulses are easy to deal with since the Laplace transform of a unit impulse is simply 1. Then the first impulse is $+1$.

The second impulse is -2 shifted by one second: $-2e^{-1s}$

The third impulse is $+1$ shifted by 2 seconds: $+1e^{-2s}$.

Putting this together, the Laplace transform of the impulses is:

$$f(s) = 1 - 2e^{-s} + e^{-2s}$$

The Laplace transform of the triangle pulse is the double integral of this:

$$f(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$

A similar technique, the single integration of a series of impulses, could be used to create a stepped waveform.
6 Appendices

6.1 Measurement Technique

Low cost oscilloscopes and waveform generators that are hosted by a personal computer, are now available to students [14]. As demonstrated in the exercises of this paper, it is possible for students to explore waveform processing concepts independently of the traditional engineering electronics laboratory. The hardware and computer host can provide powerful analysis and display functions that support learning in this environment.

For example, Syscomp instruments support the following capabilities:

- frequency-precise waveform generation and measurement
- arbitrary waveform construction and generation
- manual or automatic frequency sweep over a range of six decades
- digital cursor readout of frequency and amplitude
- spectrum analysis
- waveform math (eg. multiplication of waveforms)
- waveform image capture for reports
- waveform data capture for post-processing in spreadsheets
- vector network analysis (Bode Plots) of AC circuits

Access to this technology allows students to demonstrate and study the results of circuit analysis and design. This leads to better mastery of the theoretical concepts and facility with electronic instrumentation.

6.2 A Simple Introduction to Complex Notation

Complex numbers [15], [16], [17] and imaginary numbers are routinely used in many fields of engineering. This is especially true in electrical engineering, where they form the basis of AC circuit analysis.

AC signals can be represented as rotating vectors. For the purpose of AC circuit analysis, we take a snapshot at some instant of time, and then compare a static picture of these vectors.

Rectangular Notation

A two-dimensional vector \( Z \) may be represented by its horizontal and vertical components as a complex number (figure 29). When \( Z \) is a vector we could write:

\[
Z = x + jy
\]

where \( x \) is the horizontal component, said to be the real part. \( y \) is the vertical component, referred to as the imaginary part. \( j \) is the imaginary operator, \( \sqrt{-1} \). (Electrical engineers use \( j \) for the imaginary operator, mathematicians use \( i \)).

When a complex number is expressed like this as the sum of a real part and imaginary part, then it is said to be in rectangular notation. (The \( x \) component and the \( y \) component lie along the sides of an imaginary rectangle.)

The terms complex and imaginary should be taken with a grain of salt. At the time of their invention, the mathematical ideas presented difficulty, and these terms reflect that heritage.
For our purposes, a complex number might be better called a *vector number*. The imaginary operator, which poses such conceptual difficulty on first encounter, can be treated as a device that rotates a vector by 90°. It could be called a *quadrature rotation operator*, the term *quadrature* implying *right angle*.

Why use \( j \) as an operator on the vertical component? Because it leads to a consistent system of mathematical rules regarding operations such as addition and multiplication on the vectors. These rules are surprisingly consistent with familiar mathematical operations, as illustrated by the following example:

Refer to figure 30 and consider the vector 5 pointing along the positive X axis. This may be regarded as \( 5j^0 = 5 \times 1 = 5 \), a vector 5 units long with no rotations applied.

Then \( 5j^1 = 5j \) is the same vector rotated CCW 90° to lie along the positive Y axis.

Now consider \( 5j^2 \). This time vector is further rotated CCW by 90° so it points along the negative X axis. Then the original vector has two rotations and in complex notation becomes \( 5j^2 \). But \( j^2 = (\sqrt{-1})^2 = -1 \) so \( 5j^2 = -5 \), which is the cartesian graph notation for lying along the negative X axis. In other words, treating \( j = \sqrt{-1} \) works out correctly with the standard rules of arithmetic.

**Complex Number Arithmetic: Addition and Subtraction**

The addition and subtraction of complex numbers is quite straightforward when the numbers are expressed in rectangular notation. One simply adds the corresponding parts: real+real, imaginary+imaginary. For example:

\[
(x + jy) + (a + jb) = (x + a) + j(y + b)
\]

Multiplication and division are not so simple. It is possible to perform these operations in rectangular notation, but an easier approach is *polar* notation.

**Polar Notation**

The rectangular form of the complex number may be written as follows:

\[
x + jy = R \cos \theta + jR \sin \theta \\
= R (\cos \theta + j \sin \theta)
\]  

(80)

where:

- \( R \) is the length of the vector: \( R = \sqrt{x^2 + y^2} \)
- \( \theta \) is the angle to the vector measured from the horizontal axis, \( \tan^{-1}(y/x) \)

However, by Euler’s Identity

\[
e^{j\theta} = \cos \theta + j \sin \theta
\]  

(81)

Then equation 80 can be written as:

\[
x + jy = Re^{j\theta}
\]  

(82)

This is the *polar* form of complex notation. It is so useful in electrical engineering that many scientific calculators have built-in keystrokes to convert from rectangular to polar notation and vice-versa.

**Complex Number Arithmetic: Multiplication and Division**

The multiplication and division of complex numbers is straightforward when the numbers are expressed in polar notation. One simply multiplies the magnitude and adds the angles. For example:
\[ Z e^{j\theta} \times W e^{j\alpha} = Z \cdot W e^{j(\theta + \alpha)} \] (83)

\[ \frac{Z e^{j\theta}}{W e^{j\alpha}} = \frac{Z}{W} e^{j(\theta - \alpha)} \] (84)

\[ \frac{1}{W e^{j\alpha}} = \frac{1}{W} e^{j(0 - \alpha)} = \frac{1}{W} e^{-j\alpha} \] (85)

These operations are so common in electrical engineering that they have a shorthand form, where \( e^{j(\text{something})} \) is replaced by the angle symbol \( \angle(\text{something}) \). Then examples 83 to 85 would be written as:

\[ Z \angle\theta \times W \angle\alpha = Z \cdot W \angle(\theta + \alpha) \] (86)

\[ \frac{Z \angle\theta}{W \angle\alpha} = \frac{Z}{W} \angle(\theta - \alpha) \] (87)

\[ \frac{1}{W \angle\alpha} = \frac{1}{W} \angle - \alpha \] (88)

### 6.3 Gnuplot Magnitude and Phase Plotting Routines

The Linux version of the Gnuplot plotting program was used to create the magnitude and frequency response plots in this paper. Gnuplot is also available to run under Windows operating systems.

The lines `set terminal fig` and `set output "mag-single-tc-lp.fig"` set up the program to dump its output into a data file suitable for the XFig drawing program\(^1\).

On a Linux system, the line `set terminal x11` directs the output to the display. Under Windows, you may need to choose some other terminal. Check the Gnuplot manual under "terminal".

For reasons known only to Gnuplot, directing the output to the display with the `set terminal x11` command has the effect of disabling the output to the Xfig formatted file, so you can’t do both at once. Start with the display enabled so you can see the result. When you have a satisfactory display, you can create an XFig file (or some other format) by commenting out the lines `set terminal x11` and `replot` with a hash mark at the start, as shown in the phase plot listing below.

#### 6.3.1 First Order Lowpass Amplitude Response

```plaintext
# List of commands to plot 1st order lowpass filter magnitude response
set terminal fig
set output "mag-single-tc-lp.fig"
set ylabel "Amplitude, db"
set xlabel "Frequency, \( \omega/\omega_0 \)"
set grid
set nologscale y
set logscale x
plot [0.01:100] (10.0/2.302)*log(1/(1+(x)**2)) title '' with lines
# Factor 2.302 converts natural log to log-base-10
set terminal x11
replot
pause -1
```

\(^1\)XFig is a drawing program available on Linux systems. The Windows equivalent is JFig.
6.3.2 First Order Lowpass Phase Response

# List of commands to plot 1st order lowpass filter phase response
set terminal fig
set output "phase-single-tc-lp-a.fig"
set ylabel "Phase, degrees"
set xlabel "Frequency, $\omega/\omega_o$"
set grid
set nologscale y
set logscale x
plot [0.01:100] (180/3.14)*(-atan(x)) title '' with lines
# Factor 180/3.14 converts radians to degrees
# set terminal x11
# replot
pause -1

6.3.3 Second Order Lowpass Amplitude Response

# List of commands to plot 2nd order lowpass filter magnitude response
# There are three plots, each with different Q factor
set terminal fig
set output "mag-quadratic-lp.fig"
set ylabel "Amplitude, db"
set xlabel "Frequency, $\omega/\omega_o$"
set grid
set nologscale y
set logscale x
plot [0.05:50] \
-(10.0/2.302)*log(((1-(x**2))**2)+((2*0.25*x)**2)) title '' with lines,\ 
-(10.0/2.302)*log(((1-(x**2))**2)+((2*0.5*x)**2)) title '' with lines,\ 
-(10.0/2.302)*log(((1-(x**2))**2)+((2*1*x)**2)) title '' with lines
set terminal x11
replot
pause -1

6.3.4 Second Order Lowpass Phase Response

# List of commands to plot 2nd order lowpass filter phase response
# Getting this to work was tricky. The actual function to plot
# vs angle is atan((x/1)/(1-x**2)). However, this takes the function
# into a quadrant with a negative x axis and positive y axis,
# and so the angle jumps as it passes through +90 degrees,
# to -90 degrees and then decreases back to zero. We want a function
# that smoothly decreases from 0 to -180 degrees. To get this
# function into the right quadrant, I inverted the expression
# and then added -90 degrees.
set terminal fig
set output "phase-quadratic-q-lp.fig"
set ylabel "Phase, degrees"
set xlabel "Frequency, $\omega/\omega_o$"
#set key 6.2, 250
set grid
set nologscale y
# set angles degrees
# set yrange [-180:0]

38
set logscale x
set angles degrees
plot [0.05:50] -90+atan((1-x**2)/(x/1)) title '' with lines,
   -90+atan((1-x**2)/(x/2)) title '' with lines,
   -90+atan((1-x**2)/(x/4)) title '' with lines
set terminal x11
replot
pause -1

6.3.5 Second-Order RC Lowpass ($Q = 1/3$) Impulse Response

# List of commands to plot impulse response of 2nd-order lowpass filter
# For explanation, see 'laplace-cookbook' paper

set terminal fig # direct graphical output to a file
set output "impulse-curve.fig" # establish name of the file
set xlabel "Time Constants" # label x axis
# set ylabel "Charge\Fraction" # label y axis
# set key 6.2, 250 # position graph title
set grid # turn grid on
plot [0:10] (2.71818**(-1.52*x))*(sinh(1.13*x))
set term x11 # show output on terminal
replot # redo the plot
pause -1

6.4 Laplace Transform of Unit Impulse

The Laplace transform of some function $f(t)$ is given by:

$$\mathcal{L}f(t) = \int_0^\infty e^{-st}f(t)dt$$

Referring to figure 14(b) on page 20, during the interval $t = 0$ to $t = W$ seconds the unit impulse time-domain function is:

$$f_\delta(t) = \frac{1}{W}$$

The impulse function is zero outside the region between $t = 0$ to $t = W$, so these can become the limits of the integration. Then:

$$\mathcal{L}f_\delta = \int_0^W e^{-st} \left( \frac{1}{W} \right) dt$$

$$= \frac{1}{W} \int_0^W e^{-st} dt$$

Use the identity

$$\int e^{ax} = \frac{1}{a}e^{ax}$$

to expand equation 91:

\[^{16}\text{The approach in this section is based on [6]}\]
The final step is to make the width of the pulse $W$ approach zero. Unfortunately, this puts zero in the numerator and denominator, which makes the expression indeterminate. However, because the numerator and denominator are both zero, L'Hôpital’s Rule [7] allows us to replace both numerator and denominator by their derivatives. Then

$$
\lim_{W \to 0} \frac{1 - e^{-Ws}}{Ws} = \lim_{W \to 0} \frac{0 - We^{-Ws}}{W} = \lim_{W \to 0} e^{-Ws} = 1
$$

(94)

QED: the Laplace transform of the unit impulse is 1.

### 6.5 Laplace Transform of the Unit Step

The unit step, known as $u(t)$ has a value of unity for all times greater than zero. Plugging that into the definition of the Laplace transform (equation 89 above), we have:

$$\mathcal{L} f(t) = \int_{0}^{\infty} e^{-st} u(t) dt$$

$$= \int_{0}^{\infty} e^{-st} \cdot 1 \, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_{0}^{\infty}$$

$$= -\frac{1}{s} [e^{-\infty s} - e^{-0s}]$$

$$= -\frac{1}{s} [0 - 1]$$

$$= \frac{1}{s}
$$

That’s it!

### 7 Further Reading

Books on electric circuit analysis and signal processing usually have a section on the Laplace transform. These tend to be math-based expositions and not particularly intuitive.

Fourier analysis can be understood as a correlation process that searches the original signal for its sine and cosine (or in-phase and quadrature) components. The Laplace transform is more difficult to put on that sort of physical intuitive basis. Two sources that try are Smith [8] and Lyons [9].

Smith’s work is available on the internet, and is more detailed and extended. Lyon’s text is available as a Low Price Edition through Abe Books [10]. Both books are excellent for studies in digital signal processing.

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[17]: L'Hôpital’s Rule also applies if the numerator and denominator approach infinity.
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